421. Proposed by E. H. MOORE, The University of Chicago.

Given n continuous real-valued functions $\phi_g(x)$ $(g = 1, 2, \cdots, n)$ of the real variable x on the interval (01) and set exp. $\int_0^1 \phi_g(x)\phi_h(x) = w_{gh}$ $(g, h = 1, 2, \cdots, n)$. Prove that the determinant $| w_{gh} |$ of the matrix (w_{gh}) is always $\geqq 0$ and that it is > 0 if no two of the functions $\phi_1, \cdots, \phi_n$ are identically equal on (01).

SOLUTION BY C. F. GUMMER, Kingston, Ont.

(1) *Proof that* $| w_{gh} | \geqq 0$.

$$w_{gh} = \lim_{m=\infty} w_{gh}{}^{(m)}, \qquad \text{where} \qquad w_{gh}{}^{(m)} = 1/m \sum_{i=1}^{m} \phi_g(i/m)\phi_h(i/m);$$

and since $| w_{gh} |$ is a continuous function of the w's, it follows that

$$| w_{gh} | = | \lim w_{gh}{}^{(m)} | = \lim | w_{gh}{}^{(m)} | .$$

Now

$$| w_{gh}{}^{(m)} | = 1/m^n \, | \Sigma\phi_g(i/m)\phi_h(i/m) | ,$$

which, by the rule for a minor of a product matrix, is equal to $(1/m^n) \times$ the sum of the squares of the nth order determinants of the matrix $(\phi_g(i/m))_{\substack{i=1,\ldots m.\\ g=1,\ldots n}}$

$$\therefore \quad | w_{gh}{}^{(m)} | \geqq 0.$$

$$\therefore \quad | w_{gh} | = \lim | w_{gh}{}^{(m)} | \geqq 0.$$

(2) *The condition that* $| w_{gh} | = 0$.

Suppose $| w_{gh} | = 0$. Then there is a real linear relation

$$\sum_{g=1}^{n} a_g w_{gh} = 0 \qquad (h = 1, 2, \cdots, n). \tag{A}$$

$$\therefore \quad \int_0^1 \{\Sigma a_g\phi_g(x)\}\phi_h(x)dx = 0 \qquad (h = 1, \cdots n).$$

Multiplying by a_h and adding, we have

$$\int_0^1 \{\Sigma a_g\phi_g(x)\}^2 dx = 0.$$

Hence, the integrand being continuous,

$$\Sigma a_g\phi_g(x) = 0 \text{ on } (01). \tag{B}$$

Conversely, if (B) is true, so is (A), and $| w_{gh} | = 0$. $\therefore$ the necessary and sufficient condition for the vanishing of $| w_{gh} |$ is that the ϕ's be linearly dependent, and the problem as stated is incorrect. (Consider for example the case $\phi_1 = 1$, $\phi_2 = x$, $\phi_3 = 1 + x$.)

MECHANICS.

330. Proposed by PAUL CAPRON, U. S. Naval Academy.

A Barker's Mill operates under a head of h ft.; the linear speed of the orifices is $u(f/s)$, the speed of the water relative to the orifices is $v(f/s)$, the coefficient of discharge is c, so that $v^2 = c^2(2gh + u^2)$. Given that the work done by the water on the mill is $u/g(v - u)$ ft. lbs. per sec. per lb. of water used, find the values of u and v such that the water-power may be most economically used, and find what part of the power is so used.

SOLUTION BY THE PROPOSER.

It is required to make $f(u) = u(c\sqrt{k^2 + u^2} - u)$ a maximum. $(k^2 = 2gh)$.

$$f'(u) = \frac{cu^2}{\sqrt{k^2 + u^2}} + c\sqrt{k^2 + u^2} - 2u = 0$$

when $c(u^2 + k^2 + u^2) = 2u\sqrt{k^2 + u^2}$, or $4(1 - c^2)(u^4 + k^2u^2) = c^2k^4$.

Whence,

$$u^2 = \frac{k^2}{2}\left(\frac{1-\sqrt{1-c^2}}{\sqrt{1-c^2}}\right),$$

and

$$v^2 = c^2(k^2+u^2) = \frac{c^2k^2}{2}\left(\frac{1+\sqrt{1-c^2}}{\sqrt{1-c^2}}\right).$$

Hence,

$$u = \frac{k}{2\sqrt[4]{1-c^2}}(\sqrt{1+c}-\sqrt{1-c}), \qquad v = \frac{ck}{2\sqrt[4]{1-c^2}}(\sqrt{1+c}+\sqrt{1-c}),$$

and

$$u(v-u) = \frac{k^2}{2}(1-\sqrt{1-c^2}).$$

The available work is h ft. lbs. per sec. per lb. of water used; the work utilized is

$$\frac{u}{g}(v-u) = h(1-\sqrt{1-c^2});$$

the proportion used is $(1-\sqrt{1-c^2})$.

If we let $c = \sin\alpha$, we have $u^2 = gh\tan\alpha\tan\alpha/2$, $v = 2u\cos^2\alpha/2$, and efficiency $=$ vers α.

331. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A cyclist is riding due west at a speed of 12 miles per hr., and the wind is at the same time blowing from the southeast with a speed of $5\frac{1}{2}$ miles per hour. If the cyclist carries a small flag, in what direction will this flag fly? At what speed would the cyclist need to ride if the flag is to fly due north?

SOLUTION BY B. J. BROWN, Victor, Colo.

The component of the wind north is $11\sqrt{2}/4$ miles per hr., while that west is $11\sqrt{2}/4$ miles per hr. The resistance offered to the machine traveling west is $[12-(11\sqrt{2}/4)]$ and the reaction is toward the east. Hence the forces affecting the flag are $[12-(11\sqrt{2}/4)]$ *toward east*, and $11\sqrt{2}/4$ *toward north* and the direction θ, which the resultant makes with the east-west line $=$ arc tan $11\sqrt{2}/(48-11\sqrt{2}) = 25° 36' 55''$ N. of E. In order for the flag to fly due N. the rider must travel W. at rate of $11\sqrt{2}/4$ miles per hr. Then he counteracts the resistance E. and *only* the component north is effective.

Also solved by PAUL CAPRON, W. J. THOME, and G. W. HARTWELL.

NUMBER THEORY.

236. Proposed by V. M. SPUNAR, Chicago, Illinois.

Find integral values of x, y, z, such that

$$xy+z=\square,\quad yz+x=\square,\quad xz+y=\square.$$

NOTE BY G. H. LING, University of Saskatchewan.

The following proposition generalizes somewhat the solution of this problem which was given in the February, 1917, MONTHLY.

THEOREM. *If* (1) a *is any integer,* (2) N_1, N_2 *are integers such that* $N_1\cdot N_2 = a^2+1$, (3) $x = N_1-1$, $y = N_2-1$, $z = N_1+N_2-1+2a$, *then*

(I) $\quad xy+z=(a+1)^2;\quad yz+x=(N_2+a-1)^2;\quad zx+y=(N_1+a-1)^2,$

(II) $\quad xy+x+y=a^2;\quad yz+y+z=(N_2+a)^2;\quad zx+x+z=(N_1+a)^2.$

The solution given in the MONTHLY is the special case of this one in which

$$a = n^2+n+1.$$